Euler – Lagrange Equation

Leonhard Euler (1707–1783)

Euler was a Swiss mathematician and physicist who is widely considered the most prolific mathematician of all time. He spent much of his career in St. Petersburg, Russia, and Berlin. Even after losing his eyesight later in life, his productivity remained staggering.

Joseph-Louis Lagrange (1736–1813)

Born in Italy as Giuseppe Lodovico Lagrangia, he lived and worked primarily in Prussia and France. He was a protégé of Euler and eventually succeeded him as the Director of Mathematics at the Prussian Academy of Sciences.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

In the context of astronomical calculations, the **Euler-Lagrange equation** provides a way to determine the path of a celestial body by looking at the energy of the system rather than just the forces acting on it.

The equation is expressed as:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_{i}}\right) - \frac{\partial L}{\partial q_{i}} = 0$$

Here is how these variables function in an astronomical context:

1. The Lagrangian (L)

In orbital mechanics, L represents the difference between the **Kinetic Energy (T)** and the **Potential Energy (V)** of a planet or satellite:
$$L = T - V$$

- **T:** The energy of the planet's motion through space.
- **V:** The gravitational potential energy between the planet and the Sun (or a moon and its planet).

2. Generalized Coordinates (q_i)

In astronomy, we rarely use simple x, y, z grids. Instead, q_i refers to **coordinates** that best fit the shape of the orbit—usually **polar coordinates** (r, θ):

- **r:** The distance of the planet from the Sun.
- **θ:** The angular position of the planet in its orbit.

- **$\frac{\partial L}{\partial q_{i}}$:** This term calculates how the energy changes based on the planet's **position**. For example, gravity gets stronger as the planet moves closer to the Sun (r decreases).

3. Generalized Velocities ($\dot{q}_i$)

The "dot" over the variable signifies the derivative with respect to time, or **velocity**:

- **$\dot{r}$:** How fast the planet is moving toward or away from the Sun.
- **$\dot{\theta}$:** The angular velocity (how fast it is sweeping across the sky).
- **$\frac{\partial L}{\partial\dot{q}_{i}}$:** This represents the **conjugate momentum**. In a circular orbit, this calculation helps determine the conservation of angular momentum.

4. The Time Derivative ($\frac{d}{dt}$)

This looks at how the planet's momentum changes over **time**.

How it works for an Orbit

When you plug the energy of a planet into this equation, the result is a set of differential equations that describe the planet's trajectory.

- **The "r" calculation:** Balancing the outward "fling" of the planet's velocity against the inward pull of the Sun's gravity.
- **The "θ" calculation:** Proving that the planet sweeps out equal areas in equal times (Kepler's Second Law).

By solving for the path where the difference between kinetic and potential energy is "minimized" over time, the Euler-Lagrange equation predicts the exact elliptical path the planet must follow.

In astronomical and physical calculations involving the **Euler-Lagrange equation**, the "variable data" (like position, velocity, and energy) isn't measured as a single number by a single tool. Instead, it is derived from a chain of observations and fundamental physical constants.

Here is how those measurements are captured and converted into the variables used in the equation:

1. Measuring Position (q_i) and Velocity ($\dot{q}_i$)

To solve the equation, you first need the planet's current state. This is based on **Astrometry**:

- **Telescopic Observation:** Astronomers measure the **Right Ascension** and **Declination** (the longitude and latitude of the sky) of an object over time.
- **Parallax and Radar:** For nearby objects (like planets in our solar system), distance (r) is measured using radar bouncing or by observing the object from two different points in Earth’s orbit (parallax).
- **Deriving Velocity:** Velocity ($\dot{q}_i$) is not measured directly; it is calculated by taking the change in position over a very small interval of time (dt). If we know where Mars was yesterday and where it is today, we can calculate its velocity vector.

2. Measuring Potential Energy (V)

The potential energy variable in the Lagrangian ($L = T - V$) is based on **Gravity**:

- **The Measurement:** We measure the orbital period of moons or satellites orbiting the body.
- **The Calculation:** By observing how fast a moon orbits a planet, we can use Kepler's Third Law to "weigh" the planet (determine its mass, M).
- **The Result:** Potential energy (V) is then defined by the formula $V = -G \frac{Mm}{r}$. The "data" here is the combination of the measured mass (M), the measured distance (r), and the **Gravitational Constant (G)**.

3. Measuring Kinetic Energy (T)

Kinetic energy represents the energy of motion ($T = \frac{1}{2}mv^2$):

- **The Measurement:** This is derived directly from the velocity ($\dot{q}_i$) calculated in step 1.
- **Relativistic Adjustments:** For high-precision astronomical calculations (like the orbit of Mercury), we also use **Doppler Spectroscopy**. By measuring the "redshift" or "blueshift" of light coming from or reflecting off an object, we can measure its radial velocity with extreme precision.

4. How the "Data" enters the Equation

When you see the variables in the document—$\frac{\partial L}{\partial q_i}$ or $\frac{\partial L}{\partial \dot{q}_i}$—think of them as **sensors**:

- **$\frac{\partial L}{\partial q_i}$** acts as a measurement of the **Field**: It tells us how much the energy environment changes if the planet moves an inch to the left or right.
- **$\frac{\partial L}{\partial \dot{q}_i}$** acts as a measurement of **Momentum**: It tracks how the system's energy is tied up in its current speed.

Summary of Data Sources

Variable	Physical Basis	Measurement Tool
Position (q_i)	Geometry in Space	Optical Telescopes / Star Maps
Velocity ($\dot{q}_i$)	Displacement over Time	Sequential imaging / Doppler Shift
Mass (M)	Gravitational Pull	Orbital period observations
Energy (L)	$T - V$	Calculated from Mass, Distance, and Speed

In modern astronomy, this data is often fed into the Euler-Lagrange framework via **interferometry** and **VLBI** (Very Long Baseline Interferometry), which can measure the position of celestial objects with the precision of a hair's width seen from miles away.